

ON THE Ext -COMPUTABILITY OF SERRE QUOTIENT CATEGORIES

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ABSTRACT. To develop a constructive description of Ext in categories of coherent sheaves over certain schemes, we establish a binatural isomorphism between the Ext -groups in SERRE quotient categories \mathcal{A}/\mathcal{C} and a direct limit of Ext -groups in the ambient ABELIAN category \mathcal{A} . For Ext^1 the isomorphism follows if the thick subcategory $\mathcal{C} \subset \mathcal{A}$ is localizing. For the higher extension groups we need further assumptions on \mathcal{C} . With these categories in mind we cannot assume \mathcal{A}/\mathcal{C} to have enough projectives or injectives and therefore use YONEDA's description of Ext .

1. INTRODUCTION

Our original motivation is to develop a constructive and computer friendly description of ABELIAN categories of coherent sheaves $\mathcal{Coh} X$ on various classes of NOETHERIAN schemes X . In this setup the functors Hom and Ext^c are ubiquitous, and any constructive approach needs to incorporate these functors. For example, the global section functor on $\mathcal{Coh} X$ can be defined as $\Gamma := \text{Hom}(\mathcal{O}_X, -)$, i.e., in terms of the Hom functor and the structure sheaf \mathcal{O}_X . The higher sheaf cohomology H^i is usually defined in the non-constructive larger category¹ of quasi-coherent sheaves on X as $H^i = R^i\Gamma = \text{Ext}^i(\mathcal{O}_X, -)$.

In this paper we want to deal with computing the bivariate $\text{Ext}^i(-, -)$, where for the special univariate case of sheaf cohomology $H^i = \text{Ext}^i(\mathcal{O}_X, -)$ there often exist good algorithms. Our minimal assumption on X is that the category $\mathcal{Coh} X$ is equivalent to a SERRE quotient category $\mathcal{A}/\mathcal{C} \simeq \mathcal{Coh} X$ where \mathcal{A} is a computable category (in the sense of Appendix A) of finitely presented graded modules and $\mathcal{C} \subset \mathcal{A}$ is its thick subcategory of all modules with zero sheafification. The canonical functor $\mathcal{Q} : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{C}$ then plays the role of the exact sheafification functor $\text{Sh} : \mathcal{A} \rightarrow \mathcal{Coh} X, M \mapsto \widetilde{M}$. Some classes of schemes for which this holds are listed in [BLH12a, Section 4], including projective and toric schemes.

The computability of Ext^c would usually follow from that of Hom in case the underlying category is computable and has constructively enough projectives or enough injectives. However, as categories of *coherent* sheaves do not in general admit enough injectives or projectives we cannot assume this for the computation of Ext^c in an abstract SERRE quotient \mathcal{A}/\mathcal{C} . Hence, Ext^c in such an \mathcal{A}/\mathcal{C} cannot even be defined constructively as a derived functor using projective or injective resolutions and we are left over with YONEDA's description of Ext^c [Oor64]. Although YONEDA's description does not a priori provide

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¹Cf. [Oor64].

an algorithm to compute Ext^c , it is sufficient to prove our main result: Under certain assumptions on \mathcal{C} the computability of Ext^c in \mathcal{A}/\mathcal{C} can be reduced to the computability of Ext^c in \mathcal{A} , provided a certain (infinite) direct limit is constructive. More precisely:

Theorem 1.1. *If \mathcal{C} is a maximally almost split localizing² subcategory of an ABELIAN category \mathcal{A} then the binatural transformation³*

$$\mathcal{Q}^{\text{Ext}} : \varinjlim_{\substack{M' \leq M, \\ M/M' \in \mathcal{C}}} \text{Ext}_{\mathcal{A}}^c(M', N) \rightarrow \text{Ext}_{\mathcal{A}/\mathcal{C}}^c(M, N)$$

is an isomorphism (of ABELIAN groups) for all \mathcal{C} -torsion-free $M \in \mathcal{A}$ and \mathcal{C} -saturated $N \in \mathcal{A}$.

For applications to coherent sheaves $\mathcal{A}/\mathcal{C} \simeq \mathfrak{Coh} X$ (for X as above) we need to prove that the thick subcategory \mathcal{C} of modules with zero sheafification is maximally almost split localizing. This is for example the case if X is a projective space and \mathcal{A} is suitably chosen (see Section 3). However, it is worth mentioning that Theorem 1.1 cannot cover⁴ the case of coherent sheaves on non-smooth toric varieties. One can see this easily since the COX ring and hence the category \mathcal{A} of finitely presented graded modules over this ring is of finite global dimension while one can easily construct coherent sheaves on a non-smooth toric variety with non-vanishing Ext^c for arbitrarily high c .

The theorem suggests an algorithmic approach to the computability of Ext^c in \mathcal{A}/\mathcal{C} . To compute the left hand side $\varinjlim_{\substack{M' \leq M, \\ M/M' \in \mathcal{C}}} \text{Ext}_{\mathcal{A}}^c(M', N)$ we need to be able to compute

$\text{Ext}_{\mathcal{A}}^c$ and a direct limit of ABELIAN groups. For categories of graded modules \mathcal{A} there are well-known algorithms to compute $\text{Ext}_{\mathcal{A}}^c$. Proving that the (infinite) direct limit can be computed in finite terms depends on \mathcal{A} and \mathcal{C} . For example, in the category $\mathcal{A}/\mathcal{C} \simeq \mathfrak{Coh} X$ of coherent sheaves on a projective space the finiteness of this direct limit follows from the CASTELNUOVO-MUMFORD regularity. Thus, Theorem 1.1 is an abstract form of [Smi00, Theorem 1], without the context-specific convergence analysis. If \mathcal{A} is the category of graded modules and if the limit is reached for a certain $M' \leq M$ then one can use a graded free resolution of M' in \mathcal{A} to compute $\text{Ext}_{\mathcal{A}/\mathcal{C}}^c(M, N)$. In this case this graded free resolution of M' in \mathcal{A} corresponds, under the canonical functor $\mathcal{Q} : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{C}$, to a locally free resolution of M in $\mathcal{A}/\mathcal{C} = \mathfrak{Coh} X$ satisfying some regularity bounds. We believe that the multigraded CASTELNUOVO-MUMFORD [MS04, MS05, HSS06] can be used to prove the finiteness of the limit in the case of smooth projective toric varieties. We leave this for future work.

We briefly recall the language of SERRE quotient categories in Section 2 and deal with the $c = 0$ case of Theorem 1.1 in Section 3. In Section 4 we recall YONEDA's description of Ext^c and in Section 5 we define the binatural transformation \mathcal{Q}^{Ext} . In the main Section 6 we define the above mentioned condition which \mathcal{C} needs to satisfy and prove Theorem 1.1. There it is also proved that the theorem is valid if $c = 1$ under the weaker condition that

²Cf. Definition 6.8.

³We drop the canonical functor \mathcal{Q} in $\text{Ext}_{\mathcal{A}/\mathcal{C}}^c(\mathcal{Q}(M), \mathcal{Q}(N))$ since \mathcal{Q} is the identity on objects.

⁴contrary to Theorem 6.2.

\mathcal{C} is a localizing subcategory of \mathcal{A} (cf. Theorem 6.2). Finally, in Appendix A we briefly sketch a constructive context for this paper.

2. PRELIMINARIES ON SERRE QUOTIENTS

In this section we recall some results about SERRE quotients [Gab62]. From now on \mathcal{A} is an ABELIAN category.

A non-empty full subcategory \mathcal{C} of an ABELIAN category \mathcal{A} is called **thick** if it is closed under passing to subobjects, factor objects, and extensions. In this case the **(SERRE) quotient category** \mathcal{A}/\mathcal{C} is a category with the same objects as \mathcal{A} and Hom-groups

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) := \varinjlim_{\substack{M' \hookrightarrow M, N' \hookrightarrow N, \\ M/M', N'/N' \in \mathcal{C}}} \mathrm{Hom}_{\mathcal{A}}(M', N/N').$$

The **canonical functor** $\mathcal{Q} : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{C}$ is defined to be the identity on objects and maps a morphism $\varphi \in \mathrm{Hom}_{\mathcal{A}}(M, N)$ to its class in the direct limit $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N)$. The category \mathcal{A}/\mathcal{C} is ABELIAN and the canonical functor $\mathcal{Q} : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{C}$ is exact.

Let $\mathcal{C} \subset \mathcal{A}$ be thick. An object $M \in \mathcal{A}$ is called **\mathcal{C} -torsion-free** if M has no nonzero subobjects in \mathcal{C} . If every object $M \in \mathcal{A}$ has a **maximal \mathcal{C} -subobject** $H_{\mathcal{C}}(M)$ then we call \mathcal{C} a **thick torsion** subcategory. An object $M \in \mathcal{A}$ is called **\mathcal{C} -saturated** if it is \mathcal{C} -torsion-free and every extension of M by an object $C \in \mathcal{C}$ is trivial. Denote by $\mathrm{Sat}_{\mathcal{C}}(\mathcal{A}) \subset \mathcal{A}$ the full subcategory of \mathcal{C} -saturated objects with embedding functor $\iota : \mathrm{Sat}_{\mathcal{C}}(\mathcal{A}) \hookrightarrow \mathcal{A}$. The thick subcategory $\mathcal{C} \subset \mathcal{A}$ is called a **localizing** subcategory if the canonical functor $\mathcal{Q} : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{C}$ admits a right adjoint $\mathcal{S} : \mathcal{A}/\mathcal{C} \rightarrow \mathcal{A}$, called the **section functor** of \mathcal{Q} . The section functor $\mathcal{S} : \mathcal{A}/\mathcal{C} \rightarrow \mathcal{A}$ is left exact and preserves direct sums, the counit of the adjunction $\delta : \mathcal{Q} \circ \mathcal{S} \xrightarrow{\sim} \mathrm{Id}_{\mathcal{A}/\mathcal{C}}$ is a natural isomorphism. Let $\eta : \mathrm{Id}_{\mathcal{A}} \rightarrow \mathcal{S} \circ \mathcal{Q}$ denote the unit of the adjunction. The kernel $\ker(\eta_M : M \rightarrow (\mathcal{S} \circ \mathcal{Q})(M))$ is then the maximal \mathcal{C} -subobject $H_{\mathcal{C}}(M)$ of M . The cokernel of η_M lies in \mathcal{C} . We call $(\mathcal{S} \circ \mathcal{Q})(M)$ the **\mathcal{C} -saturation of M** . An object M in \mathcal{A} is **\mathcal{C} -saturated** if and only if η_M is an isomorphism. The image $\mathcal{S}(\mathcal{A}/\mathcal{C})$ of \mathcal{S} is a subcategory of $\mathrm{Sat}_{\mathcal{C}}(\mathcal{A})$ and the inclusion functor $\mathcal{S}(\mathcal{A}/\mathcal{C}) \hookrightarrow \mathrm{Sat}_{\mathcal{C}}(\mathcal{A})$ is an equivalence of categories with the restricted-corestricted monad $\mathcal{S} \circ \mathcal{Q} : \mathrm{Sat}_{\mathcal{C}}(\mathcal{A}) \rightarrow \mathcal{S}(\mathcal{A}/\mathcal{C})$ as a quasi-inverse. The restricted canonical functor $\mathcal{Q} : \mathrm{Sat}_{\mathcal{C}}(\mathcal{A}) \rightarrow \mathcal{A}/\mathcal{C}$ and the corestricted section functor $\mathcal{S} : \mathcal{A}/\mathcal{C} \rightarrow \mathrm{Sat}_{\mathcal{C}}(\mathcal{A})$ are quasi-inverse equivalences of categories. In particular, $\mathrm{Sat}_{\mathcal{C}}(\mathcal{A}) \simeq \mathcal{S}(\mathcal{A}/\mathcal{C}) \simeq \mathcal{A}/\mathcal{C}$ is an ABELIAN category. Define the reflector⁵ $\widehat{\mathcal{Q}} := \mathrm{cores}_{\mathrm{Sat}_{\mathcal{C}}(\mathcal{A})}(\mathcal{S} \circ \mathcal{Q}) : \mathcal{A} \rightarrow \mathrm{Sat}_{\mathcal{C}}(\mathcal{A})$. The adjunction $\widehat{\mathcal{Q}} \dashv \iota : \mathrm{Sat}_{\mathcal{C}}(\mathcal{A}) \hookrightarrow \mathcal{A}$ corresponds under the above equivalence to the adjunction $\mathcal{Q} \dashv \mathcal{S} : \mathcal{A}/\mathcal{C} \rightarrow \mathcal{A}$. They both share the same adjunction monad $\mathcal{S} \circ \mathcal{Q} = \iota \circ \widehat{\mathcal{Q}} : \mathcal{A} \rightarrow \mathcal{A}$. In particular, the reflector $\widehat{\mathcal{Q}}$ is exact and ι is left exact. $\mathcal{S}(\mathcal{A}/\mathcal{C}) \simeq \mathrm{Sat}_{\mathcal{C}}(\mathcal{A})$ are not in general ABELIAN subcategories of \mathcal{A} , as short exact sequences in $\mathrm{Sat}_{\mathcal{C}}(\mathcal{A})$ are not necessarily exact in \mathcal{A} . For more details see [BLH12b].

⁵A functor is called a reflector if it has a fully faithful right adjoint (cf. [BLH12b, 2.10]).

3. THE $c = 0$ CASE

If the thick subcategory $\mathcal{C} \subset \mathcal{A}$ is torsion then the double direct limit in the definition of the Hom-groups in \mathcal{A}/\mathcal{C} simplifies to a single direct limit

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) = \varinjlim_{\substack{M' \hookrightarrow M \\ M/M' \in \mathcal{C}}} \mathrm{Hom}_{\mathcal{A}}(M', N/H_{\mathcal{C}}(N)).$$

If furthermore $\mathcal{C} \subset \mathcal{A}$ is localizing then the Hom-adjunction⁶ between \mathcal{Q} and \mathcal{S} yields

$$(\mathrm{Hom}) \quad \mathrm{Hom}_{\mathcal{A}}(M, (\mathcal{S} \circ \mathcal{Q})(N)) \cong \mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N),$$

for all $M, N \in \mathcal{A}$, avoiding the direct limit completely. Theorem 1.1 generalizes this last formula, being the $c = 0$ case.

The monad $\mathcal{S} \circ \mathcal{Q}$ together with its unit are constructive in the case $\mathcal{A}/\mathcal{C} \simeq \mathbf{Coh} \mathbb{P}_k^n$, i.e., of coherent sheaves on the projective space $X = \mathbb{P}_k^n$ over a field k . Hence, the above mentioned Hom-adjunction can be used to compute (global) Hom-groups. More precisely, let \mathcal{A} be the category of finitely presented \mathbb{Z} -graded $k[x_0, \dots, x_n]$ -modules generated in degree ≥ 0 and \mathcal{C} be the thick subcategory of finite length modules. The \mathcal{C} -saturation of an $N \in \mathcal{A}$ is the truncated module of twisted global sections, i.e., $(\mathcal{S} \circ \mathcal{Q})(N) = \bigoplus_{i \geq 0} \Gamma(\tilde{N}(i))$, where $\tilde{N} \in \mathbf{Coh} \mathbb{P}_k^n$ is the sheafification of N . For $X = \mathbb{P}_k^n$, and hence for any projective scheme, there are already several algorithms to compute the monad $\mathcal{S} \circ \mathcal{Q}$; e.g., as an ideal transform [BS98, Theorem 20.3.15], or by the BEILINSON monad [Bei78, EFS03, DE02], or by the BGG-correspondence.

Recently, PERLING [Per11] described the section functor \mathcal{S} and hence the monad $\mathcal{S} \circ \mathcal{Q}$ for a larger class of schemes, but in a (yet) nonconstructive way. A constructive description is highly desirable as it would widen the applicability of Theorem 1.1 as an algorithm to compute Ext's for further classes of smooth schemes.

4. YONEDA'S DESCRIPTION OF Ext

Since in applications to coherent sheaves the quotient category \mathcal{A}/\mathcal{C} does not have enough projectives or injectives we use YONEDA's description of Ext^c (cf. [ML95, Section III.5]).

So let \mathcal{B} be an ABELIAN category. A c -cocycle in the $\mathrm{Ext}_{\mathcal{B}}^c(M, N)$ group ($c > 0$) is an equivalence class of c -extensions of M by N , i.e., exact \mathcal{B} -sequences

$$e : 0 \leftarrow M \leftarrow G_c \leftarrow \cdots \leftarrow G_1 \leftarrow N \leftarrow 0$$

of length $c + 2$. Two c -extensions e, e' of M by N are in directed relation if there exists a chain morphism of the form

$$\begin{array}{ccccccccccc} e : & 0 & \leftarrow & M & \leftarrow & G_c & \leftarrow & \cdots & \leftarrow & G_1 & \leftarrow & N & \leftarrow & 0 \\ & & & \parallel & & \downarrow & & & & \downarrow & & \parallel & & \\ e' : & 0 & \leftarrow & M & \leftarrow & G'_c & \leftarrow & \cdots & \leftarrow & G'_1 & \leftarrow & N & \leftarrow & 0 \end{array}$$

⁶Again, we drop the canonical functor \mathcal{Q} in $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(\mathcal{Q}(M), \mathcal{Q}(N))$ since \mathcal{Q} is the identity on objects.

For $c > 1$ this directed relation is not symmetric. A c -cocycle is an equivalence class of the equivalence relation generated by this directed relation. Abusing the notation we will denote by e the c -cocycle in $\text{Ext}_{\mathcal{B}}^c(M, N)$ of a c -extension e of M by N .

We now recall the definition of the YONEDA composition $\text{Ext}_{\mathcal{B}}^c(M, N) \times \text{Ext}_{\mathcal{B}}^{c'}(N, L) \rightarrow \text{Ext}_{\mathcal{B}}^{c+c'}(M, L)$. We start with the case $c, c' > 0$. For $e_N^M \in \text{Ext}_{\mathcal{B}}^c(M, N)$ and $e_L^N \in \text{Ext}_{\mathcal{B}}^{c'}(N, L)$ represented by the extensions

$$e_N^M : 0 \leftarrow M \leftarrow G_c \leftarrow \cdots \leftarrow G_1 \leftarrow N \leftarrow 0 \text{ and } e_L^N : 0 \leftarrow N \leftarrow G'_{c'} \leftarrow \cdots \leftarrow G'_1 \leftarrow L \leftarrow 0,$$

respectively. The YONEDA composite $e_L^M = e_N^M e_L^N \in \text{Ext}_{\mathcal{B}}^{c+c'}(M, L)$ is the $(c + c')$ -cocycle represented by the $(c + c')$ -extension

$$e_L^M : 0 \leftarrow M \leftarrow G_c \leftarrow \cdots \leftarrow G_1 \leftarrow G'_{c'} \leftarrow \cdots \leftarrow G'_1 \leftarrow L \leftarrow 0$$

of M by L , where the morphism $G_1 \leftarrow G'_{c'}$ is the composition $G_1 \leftarrow N \leftarrow G'_{c'}$.

For $c = 0$ and $c' > 0$ let $\varphi_N^M \in \text{Hom}_{\mathcal{B}}(M, N)$ and $e_L^N \in \text{Ext}_{\mathcal{B}}^{c'}(N, L)$ as above. The YONEDA composite $\varphi_N^M e_L^N \in \text{Ext}_{\mathcal{B}}^{c'}(M, L)$ is given by the pullback c' -extension

$$\begin{array}{ccccccc} \varphi_N^M e_L^N : 0 & \leftarrow & M & \leftarrow & G_{c'} & \leftarrow & G'_{c'-1} \leftarrow \cdots \leftarrow G'_1 \leftarrow L \leftarrow 0 \\ & & \varphi \downarrow & & \downarrow & & \parallel & & \parallel & & \parallel \\ e_L^N : 0 & \leftarrow & N & \leftarrow & G'_{c'} & \leftarrow & G'_{c'-1} \leftarrow \cdots \leftarrow G'_1 \leftarrow L \leftarrow 0 \end{array}$$

For $c > 0$ and $c' = 0$ let $e_N^M \in \text{Ext}_{\mathcal{B}}^c(M, N)$ as above and $\psi_N^M \in \text{Hom}_{\mathcal{B}}(N, L)$. The YONEDA composite $e_N^M \psi_L^N \in \text{Ext}_{\mathcal{B}}^c(M, L)$ is given by the pushout c -extension

$$\begin{array}{ccccccc} e_N^M : 0 & \leftarrow & M & \leftarrow & G_c & \leftarrow & \cdots \leftarrow G_2 \leftarrow G_1 \leftarrow N \leftarrow 0 \\ & & \parallel & & \parallel & & \parallel & & \downarrow & & \downarrow \psi \\ e_N^M \psi_L^N : 0 & \leftarrow & M & \leftarrow & G_c & \leftarrow & \cdots \leftarrow G_2 \leftarrow G'_1 \leftarrow L \leftarrow 0 \end{array}$$

For more details see, e.g., [HS97, Section IV.9], [BB08, Appendix B].

5. THE BINATURAL TRANSFORMATION

Let \mathcal{A} is an ABELIAN category and $\mathcal{C} \subset \mathcal{A}$ a thick subcategory. Applying the exact canonical functor $\mathcal{Q} : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{C}$ to a cocycle $e \in \text{Ext}_{\mathcal{A}}^c(M, N)$ we obtain a cocycle $\mathcal{Q}(e)$ in $\text{Ext}_{\mathcal{A}/\mathcal{C}}^c(M, N)$. In other words, the canonical functor $\mathcal{Q} : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{C}$ induces natural maps

$$\text{Ext}_{\mathcal{A}}^c(M', N/N') \rightarrow \text{Ext}_{\mathcal{A}/\mathcal{C}}^c(M, N)$$

for all $M, N \in \mathcal{A}$, $M' \leq M$, $N' \leq N$ with $M/M' \in \mathcal{C}$ and $N' \in \mathcal{C}$. For $M'' \leq M'$ with $M'/M'' \in \mathcal{C}$ and $N'' \geq N'$ with $N''/N' \in \mathcal{C}$ the cocycle

$$e' : 0 \leftarrow M' \leftarrow G'_c \leftarrow G'_{c-1} \leftarrow \cdots \leftarrow G'_2 \leftarrow G'_1 \leftarrow N/N' \leftarrow 0 \in \text{Ext}_{\mathcal{A}}^c(M', N/N')$$

⁷We write $\text{Ext}_{\mathcal{A}/\mathcal{C}}^c(M, N)$ for $\text{Ext}_{\mathcal{A}/\mathcal{C}}^c(\mathcal{Q}(M), \mathcal{Q}(N))$ since \mathcal{Q} is the identity on objects.

induces a cocycle

$$e'' : 0 \leftarrow M'' \leftarrow G_c'' \leftarrow G_{c-1}' \leftarrow \cdots \leftarrow G_2' \leftarrow G_1'' \leftarrow N/N'' \leftarrow 0 \in \text{Ext}_{\mathcal{A}}^c(M'', N/N''),$$

as the YONEDA composite $e'' = (M'' \hookrightarrow M')e'(N' \hookrightarrow N'')$. Hence, \mathcal{Q} induces a map

$$\mathcal{Q}^{\text{Ext}} : \varinjlim_{\substack{M' \leq M, N' \leq N, \\ M/M' \in \mathcal{C}, N'/N' \in \mathcal{C}}} \text{Ext}_{\mathcal{A}}^c(M', N/N') \rightarrow \text{Ext}_{\mathcal{A}/\mathcal{C}}^c(M, N)$$

for all $M, N \in \mathcal{A}$.

If $N \in \mathcal{A}$ is \mathcal{C} -saturated, then the above double limit simplifies to

$$\mathcal{Q}^{\text{Ext}} : \varinjlim_{\substack{M' \leq M, \\ M/M' \in \mathcal{C}}} \text{Ext}_{\mathcal{A}}^c(M', N) \rightarrow \text{Ext}_{\mathcal{A}/\mathcal{C}}^c(M, N)$$

for all $M \in \mathcal{A}$, as there are no non-trivial \mathcal{C} -subobjects $N' \leq N$.

Now we consider the functoriality of the left hand side

$$F : (M, N) \mapsto \varinjlim_{\substack{M' \leq M, \\ M/M' \in \mathcal{C}}} \text{Ext}_{\mathcal{A}}^c(M', N).$$

To describe the functoriality in the first argument let $\varphi : M \rightarrow L$ be an \mathcal{A} -morphism, $N \in \mathcal{A}$ (\mathcal{C} -saturated), $G^L = G_N^L \in \text{Ext}_{\mathcal{A}}^c(L, N)$, and $G^M = G_N^M = \varphi G_N^L \in \text{Ext}_{\mathcal{A}}^c(M, N)$, the YONEDA composition of φ and G^L (by construction we have that $G_{c'}^M = G_{c'}^L$ for $c' \leq c-1$). Taking the pullback of a subobject $\iota_{L'} : L' \hookrightarrow L$ with $L/L' \in \mathcal{C}$ we obtain a subobject $\iota_{M'} : M' \hookrightarrow M$ with $M/M' \in \mathcal{C}$. Sending the cocycle $\iota_{L'} G^L \in \text{Ext}_{\mathcal{A}}^c(L', N)$ to $\iota_{M'} G^M \in \text{Ext}_{\mathcal{A}}^c(M', N)$ defines the first argument action of F on φ . The proof of functoriality in the first argument follows from the identity $\iota_{M'} G^M = \iota_{M'} \varphi G^L = \varphi|_{M'} \iota_{L'} G^L = \varphi|_{M'} G^{L'}$.

$$\begin{array}{ccccccc} 0 & \longleftarrow & M' & \longleftarrow & G_c^{M'} & & \\ & \nearrow \iota_{M'} & \downarrow \varphi|_{M'} & \nearrow & \downarrow & & \\ 0 & \longleftarrow & M & \longleftarrow & G_c^M & \longleftarrow & G_{c-1}^M \longleftarrow G_{c-2}^M \\ & \downarrow \varphi & \downarrow \iota_{L'} & \downarrow & \downarrow & \parallel & \parallel \\ 0 & \longleftarrow & L' & \longleftarrow & G_c^{L'} & & \\ & \nearrow \iota_{L'} & \downarrow & \nearrow & \downarrow & \parallel & \parallel \\ 0 & \longleftarrow & L & \longleftarrow & G_c^L & \longleftarrow & G_{c-1}^L \longleftarrow G_{c-2}^L \end{array}$$

The functoriality in the second argument is simpler. Finally, the exact functor \mathcal{Q} commutes with pullbacks, implying the binaturality of \mathcal{Q}^{Ext} .

6. THE PROOF

Our goal is to give sufficient conditions for the binatural transformation \mathcal{Q}^{Ext} to be an isomorphism. For this we assume that $\mathcal{C} \subset \mathcal{A}$ is a localizing subcategory of the ABELIAN category \mathcal{A} . Then the restricted canonical functor $\mathcal{Q} : \text{Sat}_{\mathcal{C}}(\mathcal{A}) \rightarrow \mathcal{A}/\mathcal{C}$ and the corestricted section functor $\mathcal{S} : \mathcal{A}/\mathcal{C} \rightarrow \text{Sat}_{\mathcal{C}}(\mathcal{A})$ are adjoint equivalences of categories.

Remark 6.1. We will use this equivalence to replace $\text{Ext}_{\mathcal{A}/\mathcal{C}}$ by the isomorphic $\text{Ext}_{\text{Sat}_{\mathcal{C}}(\mathcal{A})}$, the functor $\mathcal{Q} : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{C}$ by $\widehat{\mathcal{Q}} := \text{cores}_{\text{Sat}_{\mathcal{C}}(\mathcal{A})}(\mathcal{S} \circ \mathcal{Q}) : \mathcal{A} \rightarrow \text{Sat}_{\mathcal{C}}(\mathcal{A})$, and finally \mathcal{Q}^{Ext} by

$$\widehat{\mathcal{Q}}^{\text{Ext}} : \varinjlim_{\substack{M' \leq M, \\ M/M' \in \mathcal{C}}} \text{Ext}_{\mathcal{A}}^c(M', N) \rightarrow \text{Ext}_{\text{Sat}_{\mathcal{C}}(\mathcal{A})}^c(M, N).$$

For simplicity we write $\text{Ext}_{\text{Sat}_{\mathcal{C}}(\mathcal{A})}^c(M, N)$ for $\text{Ext}_{\text{Sat}_{\mathcal{C}}(\mathcal{A})}^c(\widehat{\mathcal{Q}}(M), \widehat{\mathcal{Q}}(N))$. Recall that in Theorem 1.1 we require M to be \mathcal{C} -torsion-free and N to be \mathcal{C} -saturated. Since the cokernel $(\mathcal{S} \circ \mathcal{Q})(M)/M$ of η_M lies in \mathcal{C} we can as well assume without loss of generality M to be \mathcal{C} -saturated as the limit does not distinguish between M and its saturation $(\mathcal{S} \circ \mathcal{Q})(M)$.

6.1. The proof for Ext^1 . For $c = 1$ it turns out that assuming $\mathcal{C} \subset \mathcal{A}$ to be localizing is already sufficient for $\widehat{\mathcal{Q}}^{\text{Ext}}$ to be an isomorphism.

Theorem 6.2. *If \mathcal{C} is a localizing subcategory of the ABELian category \mathcal{A} then*

$$\widehat{\mathcal{Q}}^{\text{Ext}} : \varinjlim_{\substack{M' \leq M, \\ M/M' \in \mathcal{C}}} \text{Ext}_{\mathcal{A}}^1(M', N) \rightarrow \text{Ext}_{\text{Sat}_{\mathcal{C}}(\mathcal{A})}^1(M, N) \cong \text{Ext}_{\mathcal{A}/\mathcal{C}}^1(M, N)$$

is an isomorphism (of ABELian groups) for all \mathcal{C} -saturated $M, N \in \mathcal{A}$.

Proof. Recall that a short exact sequence $e : 0 \leftarrow M \xleftarrow{\pi} E \leftarrow N \leftarrow 0$ in $\text{Sat}_{\mathcal{C}}(\mathcal{A})$ are in general only left exact⁸ in \mathcal{A} , since the embedding functor $\iota : \text{Sat}_{\mathcal{C}}(\mathcal{A}) \hookrightarrow \mathcal{A}$ is in general only left exact. The \mathcal{A} -cokernel of π lies in \mathcal{C} , i.e., for $M' := \text{im } \pi$ the sequence $0 \leftarrow M' \xleftarrow{\pi} E \leftarrow N \leftarrow 0$ is exact in \mathcal{A} and $M/M' = \text{coker } \pi \in \mathcal{C}$. This yields the preimage of e under $\widehat{\mathcal{Q}}^{\text{Ext}}$ and shows surjectivity.

For the injectivity take an exact \mathcal{A} -sequence $e : 0 \leftarrow M' \leftarrow E \xleftarrow{\varphi} N \leftarrow 0$ such that the corresponding exact $\text{Sat}_{\mathcal{C}}(\mathcal{A})$ -sequence

$$\widehat{\mathcal{Q}}(e) : 0 \leftarrow \widehat{\mathcal{Q}}(M') \leftarrow \widehat{\mathcal{Q}}(E) \xleftarrow{\widehat{\mathcal{Q}}(\varphi)} \widehat{\mathcal{Q}}(N) \leftarrow 0$$

is split, i.e., e is in the kernel of $\widehat{\mathcal{Q}}^{\text{Ext}}$. By definition of split short exact sequences, there is a $\widehat{\psi} : \widehat{\mathcal{Q}}(E) \rightarrow \widehat{\mathcal{Q}}(N)$ such that $\widehat{\psi} \circ \widehat{\mathcal{Q}}(\varphi) = \text{Id}_{\widehat{\mathcal{Q}}(N)}$. Since N is \mathcal{C} -saturated the unit η_N is an isomorphism and we can define $\psi := \eta_N^{-1} \circ \widehat{\psi} \circ \eta_E$. Note that $\eta_E \circ \varphi = \widehat{\mathcal{Q}}(\varphi) \circ \eta_N$, by the naturality of η . Then $\psi \circ \varphi = \eta_N^{-1} \circ \widehat{\psi} \circ \eta_E \circ \varphi = \eta_N^{-1} \circ \widehat{\psi} \circ \widehat{\mathcal{Q}}(\varphi) \circ \eta_N = \eta_N^{-1} \circ \text{Id}_{\widehat{\mathcal{Q}}(N)} \circ \eta_N = \text{Id}_N$ implies that e is split, i.e., zero in $\text{Ext}_{\mathcal{A}}^1(M', N)$. \square

6.2. The proof of surjectivity for higher Ext 's. For $c \geq 2$ we need further conditions on the categories \mathcal{A} and \mathcal{C} . Recall that if $\mathcal{C} \subset \mathcal{A}$ is a localizing subcategory then for every object $a \in \mathcal{A}$ there exists a maximal \mathcal{C} -subobject $H_{\mathcal{C}}(a)$.

Definition 6.3. Let \mathcal{A} be an ABELian category and $\mathcal{C} \subset \mathcal{A}$ a localizing subcategory. For an object $a \in \mathcal{A}$ we call a subobject $a^{\perp} \leq a$ an **almost \mathcal{C} -complement** if $a^{\perp} \cap H_{\mathcal{C}}(a) = 0$

⁸The notion “left exact” is tied to the convention: $e : 0 \rightarrow N \rightarrow E \xrightarrow{\pi} M \rightarrow 0$.

and $a / (H_{\mathcal{C}}(a) + a^{\perp}) \in \mathcal{C}$. We call \mathcal{C} an **almost split** localizing subcategory if for each a there exists an almost \mathcal{C} -complement a^{\perp} .

$\text{Sat}_{\mathcal{C}}(\mathcal{A})$ -complexes are not exact in \mathcal{A} , but they are **exact up to \mathcal{C} -defects**, i.e., with homologies in \mathcal{C} . The following lemma identifies the defect of such complexes with the maximal \mathcal{C} -subobjects of the cokernels of the boundary operators.

Lemma 6.4. *Let \mathcal{C} be an almost split localizing subcategory of the ABELian category \mathcal{A} . Let $G'' \xleftarrow{\psi} G \xleftarrow{\varphi} G'$ be a complex in \mathcal{A} which is exact in G up to a \mathcal{C} -defect $H = \ker \psi / \text{im } \varphi$. If G'' is \mathcal{C} -torsion-free then $H_{\mathcal{C}}(\overline{G}) = H$ where $\overline{G} := \text{coker } \varphi$.*

Proof. The defect of exactness H lies in \mathcal{C} by assumption. It coincides with the maximal subobject $H_{\mathcal{C}}(\overline{G})$ in \mathcal{C} since \overline{G}/H embeds in the \mathcal{C} -torsion-free object G'' . \square

One can replace \mathcal{C} -torsion-free \mathcal{A} -complexes having defects in \mathcal{C} with exact \mathcal{A} -complexes, which are equivalent in the following sense:

Definition 6.5. Let \mathcal{C} be a thick subcategory of the ABELian category \mathcal{A} and e an \mathcal{A} -complex. We say a subcomplex e' **equals e up to \mathcal{C} -factors** if e/e' is a complex in \mathcal{C} .

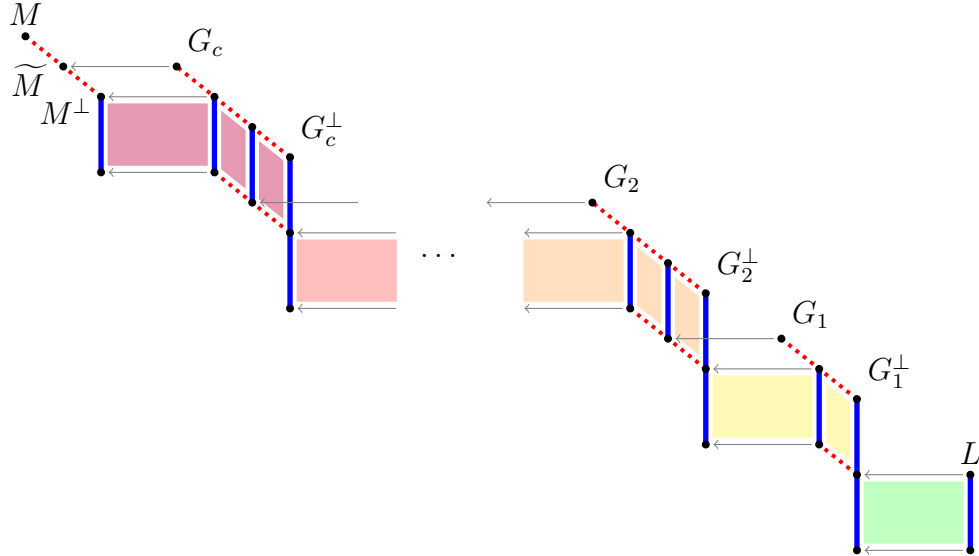
Lemma 6.6. *Let \mathcal{C} be an almost split localizing subcategory of the ABELian category \mathcal{A} and*

$$e : 0 \leftarrow M \leftarrow G_c \leftarrow \cdots \leftarrow G_1 \leftarrow L \leftarrow 0.$$

a \mathcal{C} -torsion-free \mathcal{A} -complex which is exact up to \mathcal{C} -defects. Then there exists an exact (\mathcal{C} -torsion-free) \mathcal{A} -subcomplex e^{\perp} of e

$$e^{\perp} : 0 \leftarrow M^{\perp} \leftarrow G_c^{\perp} \leftarrow \cdots \leftarrow G_1^{\perp} \leftarrow L \leftarrow 0$$

which equals e up to \mathcal{C} -factors.



Proof. Define $\overline{G}_1 := G_1 / \text{im}_{\mathcal{A}}(G_1 \hookrightarrow L)$ and the defect $H_1 := \ker(G_2 \leftarrow G_1) / \text{im}_{\mathcal{A}}(G_1 \hookrightarrow L)$. It holds that $H_1 = H_{\mathcal{C}}(\overline{G}_1)$ by Lemma 6.4. Construct the subobject $G_1^{\perp} \leq G_1$ as the

preimage in G_1 of an almost \mathcal{C} -complement of H_1 in \overline{G}_1 . Since G_1^\perp is the preimage of an almost \mathcal{C} -complement it follows that $G_1/G_1^\perp \in \mathcal{C}$.

For $i > 1$ we assume to have constructed $G_{i-1}^\perp \leq G_{i-1}$ with $G_{i-1}/G_{i-1}^\perp \in \mathcal{C}$. We proceed inductively and consider the subobject $H_i := \ker(G_{i+1} \leftarrow G_i)/\text{im}_{\mathcal{A}}(G_i \leftarrow G_{i-1} \hookrightarrow G_{i-1}^\perp)$ in the factor object $\overline{G}_i := G_i/\text{im}_{\mathcal{A}}(G_i \leftarrow G_{i-1} \hookrightarrow G_{i-1}^\perp)$. As above, $H_i = H_{\mathcal{C}}(\overline{G}_i)$ by Lemma 6.4. Again construct the subobject $G_i^\perp \leq G_i$ as the preimage in G_i of an almost \mathcal{C} -complement of H_i in \overline{G}_i . As above $G_i/G_i^\perp \in \mathcal{C}$.

Finally define the subobject $M^\perp \leq M$ as the \mathcal{A} -image $\text{im}_{\mathcal{A}}(M \leftarrow G_{c-1} \hookrightarrow G_{c-1}^\perp)$. Let $\widetilde{M} := \text{im}_{\mathcal{A}}(M \leftarrow G_{c-1})$. Then $\widetilde{M}/M^\perp \in \mathcal{C}$ as an epimorphic image of G_{c-1}/G_{c-1}^\perp under $M \leftarrow G_{c-1}$. Since also $M/\widetilde{M} \in \mathcal{C}$ it follows that $M/M^\perp \in \mathcal{C}$ as an extension of two objects in \mathcal{C} . The whole argument is visualized in the diagram⁹ below, where the dotted lines stand for (factor) objects in \mathcal{C} . \square

The above lemma yields the preimages needed to prove the surjectivity of $\widehat{\mathcal{Q}}^{\text{Ext}}$.

Proposition 6.7. *Let \mathcal{C} be an almost split localizing subcategory of the ABELian category \mathcal{A} . Then*

$$\widehat{\mathcal{Q}}^{\text{Ext}} : \varinjlim_{\substack{M' \leq M, \\ M/M' \in \mathcal{C}}} \text{Ext}_{\mathcal{A}}^c(M', N) \rightarrow \text{Ext}_{\text{Sat}_{\mathcal{C}}(\mathcal{A})}^c(M, N)$$

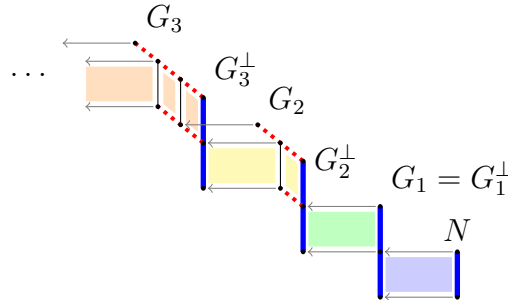
is an epimorphism (of ABELian groups) for all \mathcal{C} -saturated $M, N \in \mathcal{A}$.

Proof. For the surjectivity consider a c -extension $\widehat{e} \in \text{Ext}_{\text{Sat}_{\mathcal{C}}(\mathcal{A})}^c(M, N)$ for $c > 0$ is represented by an exact $\text{Sat}_{\mathcal{C}}(\mathcal{A})$ -complex

$$\widehat{e} : 0 \leftarrow M \leftarrow G_c \leftarrow \cdots \leftarrow G_1 \leftarrow N \leftarrow 0.$$

Now apply Lemma 6.6 to the \mathcal{A} -complex $e = \iota(\widehat{e})$ which is exact up to \mathcal{C} -defects. \square

Due to the left exactness of ι we can even choose $G_1^\perp := G_1$ in the proof of Proposition 6.7. This is illustrated by the diagram below.



⁹Cf. [Bar09] for the use of HASSE diagrams to prove statements in ABELian categories.

6.3. The proof of injectivity for higher Ext's. For the missing injectivity of $\widehat{\mathcal{Q}}^{\text{Ext}}$ we further require the existence of a (unique) maximal almost \mathcal{C} -complement. If a^\perp and $a^{\perp'}$ are almost \mathcal{C} -complements then so their sum $a^\perp + a^{\perp'} \leq a$. So, the set $\{a^\perp \mid a^\perp \text{ is an almost } \mathcal{C}\text{-complement of } a\}$ is a direct system. If the direct limit over this direct system exists then it is maximal among all almost \mathcal{C} -complements of a .

Definition 6.8. Let \mathcal{C} be a localizing subcategory of the ABELian category \mathcal{A} and $a \in \mathcal{A}$. We denote an almost \mathcal{C} -complement which is maximal among all almost \mathcal{C} -complements of $a \in \mathcal{A}$ by $H_{\mathcal{C}}^\perp(a)$ and call it, if it exists, the (unique) **maximal almost \mathcal{C} -complement (of a)**. If every object $a \in \mathcal{A}$ has a maximal almost \mathcal{C} -complement then we call \mathcal{C} a **maximally almost split** localizing subcategory of \mathcal{A} .

If \mathcal{A} is a NOETHERian ABELian category then an almost split localizing subcategory \mathcal{C} is automatically maximally almost split. In our applications to coherent sheaves \mathcal{A} will always be NOETHERian.

By Remark 6.1 the following theorem is the equivalent “saturated form” of Theorem 1.1.

Theorem 6.9. *If \mathcal{C} is a maximally almost split localizing subcategory of the ABELian category \mathcal{A} then*

$$\widehat{\mathcal{Q}}^{\text{Ext}} : \varinjlim_{\substack{M' \leq M, \\ M/M' \in \mathcal{C}}} \text{Ext}_{\mathcal{A}}^c(M', N) \rightarrow \text{Ext}_{\text{Sat}_{\mathcal{C}}(\mathcal{A})}^c(M, N)$$

is an isomorphism (of ABELian groups) for all \mathcal{C} -saturated $M, N \in \mathcal{A}$.

The idea of the proof of Theorem 6.9 is the following. Let $e \in \text{Ext}_{\mathcal{A}}^c(M'', N)$ be a representative of a c -cocycle in $\varinjlim_{\substack{M' \leq M, \\ M/M' \in \mathcal{C}}} \text{Ext}_{\mathcal{A}}^c(M', N)$. We are done if we succeed to identify the cocycle e in the direct limit $\varinjlim_{\substack{M' \leq M, \\ M/M' \in \mathcal{C}}} \text{Ext}_{\mathcal{A}}^c(M', N)$ with some “maximal” preimage $h_{\mathcal{C}}^\perp(\iota(\widehat{e}))$ of $\widehat{e} := \widehat{\mathcal{Q}}(e)$ under $\widehat{\mathcal{Q}}^{\text{Ext}}$. For this identification we need another \mathcal{C} -torsion-free \mathcal{A} -complex \widetilde{e} constructed in Proposition 6.10. Proposition 6.12 then uses the “maximality” of $h_{\mathcal{C}}^\perp(\iota(\widehat{e}))$ to relate it to \widetilde{e} .

Proposition 6.10. *Let \mathcal{C} be an almost split localizing subcategory of the ABELian category \mathcal{A} and*

$$e : 0 \leftarrow B \leftarrow G_c \leftarrow \cdots \leftarrow G_1 \leftarrow N \leftarrow 0$$

an exact \mathcal{A} -complex where B, N are \mathcal{C} -torsion-free. Then there exists an exact \mathcal{C} -torsion-free \mathcal{A} -subcomplex \widetilde{e}

$$\widetilde{e} : 0 \leftarrow \widetilde{B} \leftarrow \widetilde{G}_c \leftarrow \cdots \leftarrow \widetilde{G}_1 \leftarrow N \leftarrow 0,$$

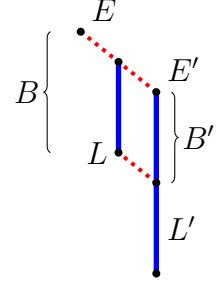
which equals e up to \mathcal{C} -factors.

For the induction proof we need the next lemma, which shows how to replace short exact sequences by short exact sequences of \mathcal{C} -torsion-free objects.

Lemma 6.11. *Let \mathcal{C} be an almost split localizing subcategory of the ABELian category \mathcal{A} and $e : 0 \leftarrow B \leftarrow E \leftarrow L \leftarrow 0$ a short exact \mathcal{A} -sequence with \mathcal{C} -torsion-free B .*

- (a) Then there exists a short exact \mathcal{A} -subsequence $e' : 0 \leftarrow B' \leftarrow E' \leftarrow L' \leftarrow 0$ with \mathcal{C} -torsion-free objects which equals e up to \mathcal{C} -factors.
- (b) If additionally L is \mathcal{C} -torsion-free then L' can be chosen to be L .

Proof. For the proof of (a) we interpret L as a subobject of E with factor object B . Let E' be an almost \mathcal{C} -complement in E . Hence E/E' lies in \mathcal{C} . Now define $L' = L \cap E'$ and $B' := E'/L'$. L' is \mathcal{C} -torsion-free as a subobject of E' and B' is \mathcal{C} -torsion-free since it is isomorphic to the subobject $B'' = (E' + L)/L$ of $B = E/L$. Finally B/B'' lies in \mathcal{C} as a factor of E/E' and L/L' lies in \mathcal{C} since it is isomorphic to the subobject $(E' + L)/E'$ of E/E' .



For the proof of (b) note that if additionally L is \mathcal{C} -torsion-free then we can choose E' to contain L . Hence $L' := L \cap E' = L$. \square

Proof of Proposition 6.10. We will construct \tilde{e} by induction on c . The case $c = 1$ is Lemma 6.11.(b). Now assume the statement is true for $c - 1$ (i.e., for complexes of length $c + 1$). Define $L := G_c / \text{im}(G_c \leftarrow G_{c-1})$. Write e as the YONEDA product (i.e., concatenation) $e_1 \circ e_2$ of the short exact \mathcal{A} -sequence $e_1 : 0 \leftarrow B \leftarrow G_c \leftarrow L \leftarrow 0$ and the exact \mathcal{A} -complex $e_2 : 0 \leftarrow L \leftarrow G_{c-1} \leftarrow \cdots \leftarrow G_1 \leftarrow N \leftarrow 0$. First apply Lemma 6.11.(a) to e_1 and obtain the short exact \mathcal{C} -torsion-free \mathcal{A} -sequence $e'_1 : 0 \leftarrow B' \leftarrow G'_c \leftarrow L' \leftarrow 0$. Now replace L in e_2 by its \mathcal{C} -torsion-free subobject L' and G_{c-1} by the (full) preimage G'_{c-1} of L' in G_{c-1} and obtain the exact \mathcal{A} -complex $e'_2 : 0 \leftarrow L' \leftarrow G'_{c-1} \leftarrow G_{c-2} \leftarrow \cdots \leftarrow G_1 \leftarrow N \leftarrow 0$. By the induction hypothesis there exists an exact \mathcal{C} -torsion-free \mathcal{A} -subcomplex $\tilde{e}'_2 : 0 \leftarrow \tilde{L} \leftarrow \tilde{G}_{c-1} \leftarrow \tilde{G}_{c-2} \leftarrow \cdots \leftarrow \tilde{G}_1 \leftarrow N \leftarrow 0$ of e'_2 . Replacing L' by its subobject \tilde{L} in e'_1 we obtain the \mathcal{C} -torsion-free \mathcal{A} -complex $\tilde{e}'_1 : 0 \leftarrow B' \leftarrow G'_c \leftarrow \tilde{L} \leftarrow 0$ which is exact up to a \mathcal{C} -defect. Now apply Lemma 6.6 to \tilde{e}'_1 and obtain the short exact \mathcal{C} -torsion-free \mathcal{A} -sequence $\tilde{e}_2 : 0 \leftarrow \tilde{B} \leftarrow \tilde{G}_c \leftarrow \tilde{L} \leftarrow 0$. Finally define $\tilde{e} := \tilde{e}_1 \circ \tilde{e}_2$.

All constructions in this proof yield subcomplexes equal to their superobjects up to \mathcal{C} -factors. Thus, we conclude the \tilde{e} equals e up to \mathcal{C} -factors. \square

For the construction of the “maximal” preimage $h_{\mathcal{C}}^{\perp}(\iota(\hat{e}))$ let \mathcal{C} be a maximally almost split localizing subcategory of the ABELian category \mathcal{A} . Let \check{e} be a \mathcal{C} -torsion-free \mathcal{A} -complex which is exact up to \mathcal{C} -defects. Denote by $h_{\mathcal{C}}^{\perp}(\check{e})$ the *unique* exact \mathcal{A} -subcomplex constructed in the proof of Lemma 6.6 where now we take all \mathcal{C} -complements to be maximal. The next proposition characterizes the “maximality” of $h_{\mathcal{C}}^{\perp}(\check{e})$ among the *exact* \mathcal{C} -torsion-free \mathcal{A} -subcomplexes of \check{e} .

Proposition 6.12. *Let \mathcal{C} be a maximally almost split localizing subcategory of the ABELian category \mathcal{A} and*

$$\check{e} : 0 \leftarrow \check{M} \leftarrow \check{G}_c \leftarrow \cdots \leftarrow \check{G}_1 \leftarrow \check{G}_0 \leftarrow 0.$$

a \mathcal{C} -torsion-free \mathcal{A} -complex which is exact up to \mathcal{C} -defects. Then any exact \mathcal{A} -subcomplex

$$\tilde{e} : 0 \leftarrow \tilde{M} \leftarrow \tilde{G}_c \leftarrow \cdots \leftarrow \tilde{G}_1 \leftarrow \tilde{G}_0 \leftarrow 0$$

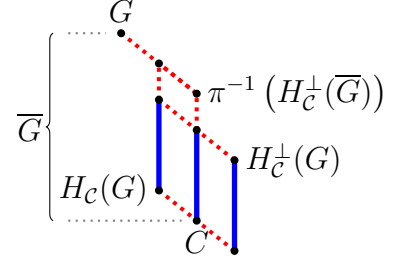
of e which equals e up to \mathcal{C} -factors is a subcomplex of

$$h_{\mathcal{C}}^{\perp}(\check{e}) : 0 \leftarrow M^{\perp} \leftarrow G_c^{\perp} \leftarrow \cdots \leftarrow G_1^{\perp} \leftarrow G_0^{\perp} \leftarrow 0.$$

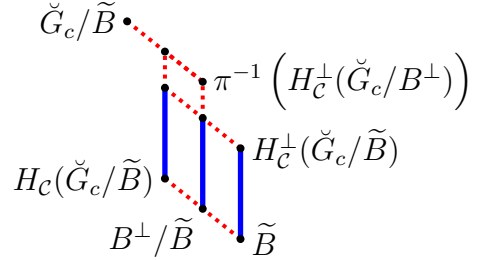
For the proof of this proposition we need a lemma.

Lemma 6.13. *Let $0 \leftarrow \overline{G} \xleftarrow{\pi} G \leftarrow C \leftarrow 0$ be a short exact \mathcal{A} -sequence with $C \in \mathcal{C}$. Then $\pi^{-1}(H_{\mathcal{C}}^{\perp}(\overline{G})) \geq H_{\mathcal{C}}^{\perp}(G)$ (for the notation $H_{\mathcal{C}}^{\perp}$ cf. Definition 6.8).*

Proof. Since $H_{\mathcal{C}}^{\perp}(G) \cap C = 0$ it follows that $\pi(H_{\mathcal{C}}^{\perp}(G))$ is isomorphic to $H_{\mathcal{C}}^{\perp}(G)$. Hence $\pi(H_{\mathcal{C}}^{\perp}(G))$ is \mathcal{C} -torsion-free and thus lies in $H_{\mathcal{C}}^{\perp}(\overline{G})$. \square



Proof of Proposition 6.12. The proof is by induction on c . For $c = 0$ the complex \check{e} consists of a single isomorphism $0 \leftarrow \check{M} \leftarrow \check{G}_0 \leftarrow 0$ and the statement is trivial. Now assume the statement is true for $c - 1 \geq 0$ (i.e., for complexes of length $c + 1$). Define the subobjects $B^{\perp} := \text{im}(\check{G}_c \leftarrow \check{G}_{c-1} \hookrightarrow G_{c-1}^{\perp})$ and $\tilde{B} := \text{im}(\check{G}_c \leftarrow \check{G}_{c-1} \hookrightarrow \tilde{G}_{c-1})$ of $\ker(\check{M} \leftarrow \check{G}_c)$. By the induction hypothesis the statement is true for $0 \leftarrow \ker(\check{M} \leftarrow \check{G}_c) \leftarrow \check{G}_{c-1} \leftarrow \cdots \leftarrow \check{G}_1 \leftarrow \check{G}_0 \leftarrow 0$. In particular $\tilde{G}_{c-1} \leq G_{c-1}^{\perp}$ implies $\tilde{B} \leq B^{\perp}$ and $B^{\perp}/\tilde{B} \in \mathcal{C}$. Note that $\ker(\check{M} \leftarrow \check{G}_c)/\tilde{B} \in \mathcal{C}$ since it is an extension of B^{\perp}/\tilde{B} with the defect $\ker(\check{M} \leftarrow \check{G}_c)/B^{\perp}$ of \check{e} , and both are in \mathcal{C} . Thus, we have $\pi^{-1}(H_{\mathcal{C}}(\check{G}_c/B^{\perp})) = H_{\mathcal{C}}(\check{G}_c/\tilde{B}) = \ker(\check{M} \leftarrow \check{G}_c)/\tilde{B}$ by Lemma 6.4. The short exact sequence $0 \leftarrow \check{G}_c/B^{\perp} \xleftarrow{\pi} \check{G}_c/\tilde{B} \leftarrow B^{\perp}/\tilde{B} \leftarrow 0$ satisfies the assumption of Lemma 6.13. Thus, $G_c^{\perp}/\tilde{B} := \pi^{-1}(H_{\mathcal{C}}^{\perp}(\check{G}_c/B^{\perp})) \geq H_{\mathcal{C}}^{\perp}(\check{G}_c/\tilde{B}) = \tilde{G}_c/\tilde{B}$ and we are done. \square



Proof of Theorem 6.9. Let $e \in \text{Ext}_{\mathcal{A}}^c(M'', N)$ be a representative of a c -cocycle in the limit $\varinjlim_{M'/M' \in \mathcal{C}} \text{Ext}_{\mathcal{A}}^c(M', N)$. The \mathcal{A} -complex $h_{\mathcal{C}}^{\perp}(\iota(\hat{e})) \in \text{Ext}_{\mathcal{A}}^c(M^{\perp}, N)$ is a preimage of $\hat{e} :=$

$\hat{\mathcal{Q}}(e) \in \text{Ext}_{\text{Sat}_{\mathcal{C}}(\mathcal{A})}^c(\hat{\mathcal{Q}}(M), \hat{\mathcal{Q}}(N))$ under $\hat{\mathcal{Q}}^{\text{Ext}}$, which we call the **maximal preimage**. We apply Proposition 6.10 to e and obtain the \mathcal{C} -torsion-free subcomplex $\tilde{e} \in \text{Ext}_{\mathcal{A}}^c(\widetilde{M''}, N)$ which equals e up to \mathcal{C} -factors. By Proposition 6.12 \tilde{e} is also an \mathcal{A} -subcomplex of $h_{\mathcal{C}}^{\perp}(\iota(\hat{e}))$ and these complexes are also equal up to \mathcal{C} -factors. Furthermore, the two chain morphisms $e \hookrightarrow \tilde{e} \hookrightarrow h_{\mathcal{C}}^{\perp}(\iota(\hat{e}))$ are the identity on N . Thus, the two chain morphisms $e \hookrightarrow \tilde{e} \hookrightarrow h_{\mathcal{C}}^{\perp}(\iota(\hat{e}))$ identify the c -cycles e and $h_{\mathcal{C}}^{\perp}(\iota(\hat{e}))$ in $\varinjlim_{M'/M' \in \mathcal{C}} \text{Ext}_{\mathcal{A}}^c(M', N)$.

Let $f \in \text{Ext}_{\mathcal{A}}^c(M''', N)$ be another representative of a c -cocycle in the above limit with $\hat{f} := \hat{\mathcal{Q}}(f)$ equivalent to \hat{e} . Then, by Lemma 6.14, $h_{\mathcal{C}}^{\perp}(\iota(\hat{e}))$ and $h_{\mathcal{C}}^{\perp}(\iota(\hat{f}))$ are equivalent in $\varinjlim_{M'/M' \in \mathcal{C}} \text{Ext}_{\mathcal{A}}^c(M', N)$, and thereby also e and f . \square

Lemma 6.14. *Let \mathcal{C} be a maximally almost split localizing subcategory of the ABELIAN category \mathcal{A} and $M, N \in \mathcal{A}$ two \mathcal{C} -saturated objects. Let \widehat{e}, \widehat{f} represent the same cocycle in $\text{Ext}_{\text{Sat}_{\mathcal{C}}(\mathcal{A})}^{\mathcal{C}}(\widehat{\mathcal{Q}}(M), \widehat{\mathcal{Q}}(N))$. Then, $h_{\mathcal{C}}^{\perp}(\iota(\widehat{e}))$ and $h_{\mathcal{C}}^{\perp}(\iota(\widehat{f}))$ represent the same cocycle in $\varinjlim_{M/M' \in \mathcal{C}} \text{Ext}_{\mathcal{A}}^{\mathcal{C}}(M', N)$.*

Proof. Without loss of generality assume that the equivalence of \widehat{e} and \widehat{f} is given by one chain morphism $\varphi : \widehat{e} \rightarrow \widehat{f}$ (being the identity on $\widehat{\mathcal{Q}}(M)$ and $\widehat{\mathcal{Q}}(N)$). The pullback of $h_{\mathcal{C}}^{\perp}(\iota(\widehat{f})) \hookrightarrow \iota(\widehat{f})$ along $\iota(\varphi)$ yields a \mathcal{C} -torsion-free subcomplex \dot{e} of $\iota(\widehat{e})$. The subcomplex \dot{e} is equal to $\iota(\widehat{e})$ up to \mathcal{C} -factors, as $\iota(\widehat{e})/\dot{e}$ is isomorphic to a subobject of $\iota(\widehat{f})/h_{\mathcal{C}}^{\perp}(\iota(\widehat{f})) \in \mathcal{C}$. In particular, $\mathcal{Q}(\dot{e} \hookrightarrow \iota(\widehat{e}))$ is a chain isomorphism in \mathcal{A}/\mathcal{C} . Hence \dot{e} is exact up to \mathcal{C} -defects, because also $\iota(\widehat{e})$ is exact up to \mathcal{C} -defects. Applying Lemma 6.6 to \dot{e} yields an exact subcomplex \dot{e}^{\perp} of $\iota(\widehat{e})$ and these complexes are again equal up to a \mathcal{C} -factor. By Proposition 6.12 we get an embedding $\dot{e} \hookrightarrow h_{\mathcal{C}}^{\perp}(\iota(\widehat{e}))$. Now, the two chain morphisms $h_{\mathcal{C}}^{\perp}(\iota(\widehat{e})) \hookleftarrow \dot{e}^{\perp} \hookrightarrow \dot{e} \hookrightarrow h_{\mathcal{C}}^{\perp}(\iota(\widehat{f}))$ identify $h_{\mathcal{C}}^{\perp}(\iota(\widehat{e}))$ and $h_{\mathcal{C}}^{\perp}(\iota(\widehat{f}))$ in $\varinjlim_{M/M' \in \mathcal{C}} \text{Ext}_{\mathcal{A}}^{\mathcal{C}}(M', N)$, where $\dot{e} \rightarrow h_{\mathcal{C}}^{\perp}(\iota(\widehat{f}))$ is the pullback morphism. \square

APPENDIX A. SKETCH OF THE PROPER CONSTRUCTIVE SETUP

We now roughly describe the constructive context of this paper. A detailed description would require a more elaborate preparation and would distract from the main result of this paper, which in this form should already be self-contained. The standard way to express mathematical notions constructively is to provide algorithms for all disjunctions and all existential quantifiers appearing in the defining axioms of a mathematical structure. In the case of ABELIAN categories this led us to the notion of a **computable ABELIAN category** [BLH11b]. Given that, all constructions which only depend on a category being ABELIAN become computable¹⁰. The computability of \mathcal{A} implies, in particular, that we can compute in its Hom-groups only *locally*, i.e., we can decide element membership in the Hom-sets, whether morphisms are zero, add and subtract morphisms, and hence decide the equality of two morphisms. This does not imply that we can “oversee” a Hom-group in any way, not even being able to decide its triviality (see Hom-computability below).

For an ABELIAN category \mathcal{A} with thick subcategory $\mathcal{C} \subset \mathcal{A}$ we prove in [BLH] that \mathcal{A}/\mathcal{C} is computable once the ABELIAN category \mathcal{A} is computable and the membership in $\mathcal{C} \subset \mathcal{A}$ is constructively decidable.

We call $\mathcal{C} \subset \mathcal{A}$ **constructively localizing** if there exists algorithms to compute the adjunction monad $\mathcal{S} \circ \mathcal{Q}$ together with its unit. Formula (Hom) in Section 3 proves that if \mathcal{A} is Hom-computable and $\mathcal{C} \subset \mathcal{A}$ is constructively localizing then \mathcal{A}/\mathcal{C} is Hom-computable,

¹⁰A constructive treatment of spectral sequences along these lines can be found in [Bar09] with a computer implementation in [BLH11a].

where **Hom-computability** means the computability as an enriched¹¹ category over a *computable* monoidal category¹².

Theorem 1.1 implies that \mathcal{A}/\mathcal{C} is Ext-computable if \mathcal{A} is Ext-computable and $\mathcal{C} \subset \mathcal{A}$ is maximally almost split localizing and constructively localizing and the direct limit is constructive. We would define **Ext-computability** to be the Hom-computability of the derived category of \mathcal{A} . This would lead too far away.

Finally, we note that the entire proof of Theorem 1.1 is constructive and suited for computer implementation. So if we assume that \mathcal{A} is computable and $\mathcal{C} \subset \mathcal{A}$ is constructively maximally almost split localizing¹³ then the proof of Theorem 1.1 provides an algorithm to compute images and preimages of elements represented as YONEDA cocycles under $\widehat{\mathcal{Q}}^{\text{Ext}} : \varinjlim_{\substack{M' \leq M, \\ M/M' \in \mathcal{C}}} \text{Ext}_{\mathcal{A}}^c(M', N) \rightarrow \text{Ext}_{\text{Sat}_{\mathcal{C}}(\mathcal{A})}^c(M, N)$. Furthermore, if \mathcal{A} is Hom-computable and has constructively enough projectives or injectives then we can decide equality of (YONEDA) cocycles (cf. [BB08, Appendix B]).

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¹¹Enriched categories are usually required be small. In an algorithmic setting any category is small, as the possible states of the computer memory is a set.

¹²... of ABELIAN groups, k -vector spaces, etc., depending on the context.

¹³I.e., constructively localizing and that we can algorithmically construct the maximal almost \mathcal{C} -complement of objects in \mathcal{A}

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